

# A note on quickly sampling a sparse matrix with low rank expectation

Karl Rohe, Jun Tao, Xintian Han, Norbert Binkiewicz

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## Abstract

Given matrices  $X, Y \in \mathbb{R}^{n \times K}$  and  $S \in \mathbb{R}^{K \times K}$  with positive elements, this paper proposes an algorithm `fastRG` to sample a sparse matrix  $A$  with low rank expectation  $E(A) = XSY^T$  and independent Poisson elements. This allows for quickly sampling from a broad class of stochastic blockmodel graphs (degree-corrected, mixed membership, overlapping) all of which are specific parameterizations of the generalized random product graph model defined in Section 2.2. The basic idea of `fastRG` is to first sample the number of edges  $m$  and then sample each edge. The key insight is that because of the low rank expectation, it is easy to sample individual edges. The naive “element-wise” algorithm requires  $O(n^2)$  operations to generate the  $n \times n$  adjacency matrix  $A$ . In sparse graphs, where  $m = O(n)$ , ignoring log terms, `fastRG` runs in time  $O(n)$ . An implementation in R is available on github. A computational experiment in Section 2.4 simulates graphs up to  $n = 10,000,000$  nodes with  $m = 100,000,000$  edges. For example, on a graph with  $n = 500,000$  and  $m = 5,000,000$ , `fastRG` runs in less than one second on a 3.5 GHz Intel i5.

## 1 Introduction

The random dot product graph (RDPG) model serves as a test bed for various types of clustering and statistical inference algorithms. This model generalizes the Stochastic Blockmodel (SBM), the Overlapping SBM, the Mixed Membership SBM, the Degree-Corrected SBM, and the latent eigenmodel which is itself a generalization of the Latent Space Model [Holland et al., 1983, Latouche and Ambroise, 2011, Airoldi et al., 2008, Karrer and Newman, 2011, Hoff, 2007, Hoff et al., 2002]. Under the RDPG, each node  $i$  has a (latent) node feature  $x_i \in \mathbb{R}^K$  and the probability that node  $i$  and  $j$  share an edge is parameterized by  $\langle x_i, x_j \rangle$  [Young and Scheinerman, 2007].

While many network analysis algorithms only require  $O(m)$  operations, where  $m$  is the number of edges in the graph, sampling RDPGs with the naive “element-wise” algorithm takes  $O(n^2)$  operations, where  $n$  is the number of nodes. In particular, sparse eigensolvers compute the leading  $k$  eigenvectors of such graphs in  $O(km)$  operations (e.g. in ARPACK [Lehoucq et al.]). As such, sampling an RDPG is a computational bottleneck in simulations to examine many network analysis algorithms.

The organization of this paper is as follows. Section 2 gives `fastRG`. Section 2.1 relates `fastRG` to `xlr`, a new class of edge-exchangeable random graphs with low-rank expectation. Section 2.2 presents a generalization of the random dot product graph. Theorem 2 shows that `fastRG` samples Poisson-edge graphs from this model. Then, Theorem 3 in Section 2.3 shows how `fastRG` can be used to approximate a certain class of Bernoulli-edge graphs. Section 2.4 describes our implementation of `fastRG` (available at <https://github.com/karlrohe/fastRG>) and assesses the empirical run time of the algorithm.

### 1.1 Notation

Let  $G = (V, E)$  be a graph with the node set  $V = \{1, \dots, n\}$  and the edge set  $E = \{(i, j) : i \text{ is connected to } j\}$ . In a directed graph, each edge is an ordered pair of nodes while in an undirected graph, each edge is an unordered pair of nodes. A multi-edge graph allows for repeated edges. In any graph, a self-loop is an edge that connects a node to itself. Let the adjacency matrix  $A \in \mathbb{R}^{n \times n}$  contain the number of edges from  $i$

to  $j$  in element  $A_{ij}$ . For column vectors  $x \in \mathbb{R}^a$ ,  $z \in \mathbb{R}^b$  and  $S \in \mathbb{R}^{a \times b}$ , define  $\langle x, z \rangle_S = x^T S z$ ; this function is not necessarily a proper inner product because it does not require that  $S$  is non-negative definite. We use standard big- $O$  and little- $o$  notations, i.e. for sequence  $x_n, y_n$ ;  $x_n = o(y_n)$  when  $y_n$  is nonzero implies  $\lim_{n \rightarrow \infty} x_n/y_n = 0$ ;  $x_n = O(y_n)$  implies there exists a positive real number  $M$  and an integer  $N$  such that  $|x_n| \leq M|y_n|$ ,  $\forall n \geq N$ .

## 2 fastRG

fastRG is motivated by the wide variety of low rank graph models that specify the expectation of the adjacency matrix as  $E(A) = X S X^T$  for some matrix (or vector)  $X$  and some matrix (or value)  $S$ .

Types of low rank models	$X \in$	In each row...
SBM	$\{0, 1\}^{n \times K}$	a single one
Degree-Corrected SBM	$\mathbb{R}^{n \times K}$	a single non-zero positive entry
Mixed Membership SBM	$\mathbb{R}^{n \times K}$	non-negative and sum to one
Overlapping SBM	$\{0, 1\}^{n \times K}$	a mix of 1s and 0s
Erdős-Rényi	$\{1\}^n$	a single one
Chung-Lu	$\mathbb{R}^n$	a single positive value

Table 1: Restrictions on the matrix  $X$  create different types of well known low rank models. There are further differences between these models that are not emphasized by this table.

Given  $X \in \mathbb{R}^{n \times K_x}$ ,  $Y \in \mathbb{R}^{d \times K_y}$  and  $S \in \mathbb{R}^{K_x \times K_y}$ , fastRG samples a random graph whose  $n \times d$  incidence matrix has expectation  $X S Y^T$ . Importantly, fastRG requires that the elements of  $X, Y$ , and  $S$  are non-negative. This condition holds for all of the low rank models in the above table. Each of those models set  $Y = X$  and enforce different restrictions on the matrix  $X$ .

As stated below, fastRG samples a (i) directed graph with (ii) multiple edges and (iii) self-loops. After sampling, these properties can be modified to create a simple graph (undirected, no repeated edges, and no self-loops); see Remarks 1 and 2 in Section 2.2 and Theorem 3 in Section 2.3.

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### Algorithm 1 fastRG( $X, S, Y$ )

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**Require:**  $X \in \mathbb{R}^{n \times K_x}$ ,  $S \in \mathbb{R}^{K_x \times K_y}$ , and  $Y \in \mathbb{R}^{d \times K_y}$  with all matrices containing non-negative entries.

Compute diagonal matrix  $C_X \in \mathbb{R}^{K_x \times K_x}$  with  $C_X = \text{diag}(\sum_i X_{i1}, \dots, \sum_i X_{iK_x})$ .

Compute diagonal matrix  $C_Y \in \mathbb{R}^{K_y \times K_y}$  with  $C_Y = \text{diag}(\sum_i Y_{i1}, \dots, \sum_i Y_{iK_y})$ .

Define  $\tilde{X} = X C_X^{-1}$ ,  $\tilde{S} = C_X S C_Y$ , and  $\tilde{Y} = Y C_Y^{-1}$ .

Sample the number of edges  $m \sim \text{Poisson}(\sum_{u,v} \tilde{S}_{uv})$ .

**for**  $\ell = 1 : m$  **do**

    Sample  $U \in \{1, \dots, K_x\}, V \in \{1, \dots, K_y\}$  with  $\mathbb{P}(U = u, V = v) \propto \tilde{S}_{uv}$ .

    Sample  $I \in \{1, \dots, n\}$  with  $\mathbb{P}(I = i) = \tilde{X}_{iU}$ .

    Sample  $J \in \{1, \dots, d\}$  with  $\mathbb{P}(J = j) = \tilde{Y}_{jV}$ .

    Add edge  $(I, J)$  to the graph, allowing for multiple edges  $(I, J)$ .

**end for**

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An implementation in R is available at <https://github.com/karlrohe/fastRG>. As discussed in Section 2.4, in order to make the algorithm more efficient, the implementation is slightly different from the statement of the algorithm above.

There are two model classes that can help to interpret the graphs generated from fastRG and those model classes are explored in the next two subsections. Throughout all of the discussion, the key fact that is exploited by fastRG is given in the next Theorem.

**Theorem 1.** Suppose that  $X \in \mathbb{R}^{n \times K_x}$ ,  $Y \in \mathbb{R}^{d \times K_y}$  and  $S \in \mathbb{R}^{K_x \times K_y}$  all contain non-negative entries. Define  $x_i \in \mathbb{R}^{K_x}$  as the  $i$ th row of  $X$ . Define  $y_j \in \mathbb{R}^{K_y}$  as the  $j$ th row of  $Y$ . Let  $(I, J)$  be a single edge sampled inside

the for loop in *fastRG*( $X, S, Y$ ), then

$$\mathbb{P}((I, J) = (i, j)) \propto \langle x_i, y_j \rangle_S.$$

*Proof.*

$$\begin{aligned} \mathbb{P}((I, J) = (i, j)) &= \sum_{u,v} \mathbb{P}((I, J) = (i, j) | (U, V) = (u, v)) \mathbb{P}((U, V) = (u, v)) \\ &= \frac{\sum_{u,v} \tilde{X}_{iu} \tilde{Y}_{jv} \tilde{S}_{uv}}{\sum_{u,v} \tilde{S}_{u,v}} = \frac{\sum_{u,v} X_{iu} Y_{jv} S_{uv}}{\sum_{u,v} S_{u,v}} = \frac{x_i^T S y_j}{\sum_{a,b} x_a^T S y_b} \end{aligned}$$

□

For notational simplicity, the rest of the paper will suppose that  $Y = X \in \mathbb{R}^{n \times K}$ .

## 2.1 *fastRG* samples from xlr: a class of edge-exchangeable random graphs

There has been recent interest in edge exchangeable graph models with blockmodel structure (e.g. Herlau et al. [2016], Todeschini and Caron [2016]). To characterize a broad class of such models, we propose xlr.

**Definition 1.** [*xlr*] An *xlr* graph on  $n$  nodes and  $K$  dimensions is generated as follows,

1. Sample  $(X, S) \in \mathbb{R}^{n \times K} \times \mathbb{R}^{K \times K}$  from some distribution and define  $x_i$  as the  $i$ th row of  $X$ .
2. Initialize the graph to be empty.
3. Add independent edges  $e_1, e_2, \dots$  to the graph, where

$$\mathbb{P}(e_\ell = (i, j)) = \frac{\langle x_i, x_j \rangle_S}{\sum_{a,b} \langle x_a, x_b \rangle_S}.$$

From Theorem 1, *fastRG* provides a way to sample the edges in xlr.

An xlr graph is both (i) edge-exchangeable as defined by Crane and Dempsey [2016] and (ii) conditional on  $X$  and  $S$ , its expected adjacency matrix is low rank. By sampling  $X$  to satisfy one set of restrictions specified in Table 1, xlr provides a way to sample edge exchangeable blockmodels. xlr stands for *edge-exchangeable and low rank* because it characterizes all edge-exchangeable and low rank random graph models on a finite number of nodes. In particular, by Theorem 4.2 in Crane and Dempsey [2016] if a random undirected graph with an infinite number of edges is edge exchangeable, then the edges are drawn iid from some randomly chosen distribution on edges  $f$ . Moreover, let  $B$  be the adjacency matrix of a single edge drawn from  $f$ . Under the assumption that  $E(B|f)$  is rank  $K$ , there exist matrices  $X \in \mathbb{R}^{n \times K}$  and  $S \in \mathbb{R}^{K \times K}$  that are a function of  $f$  and give the eigendecomposition  $E(B|f) = X S X^T$ . This implies that  $\mathbb{P}(e_1 = (i, j) | f) \propto \langle x_i, x_j \rangle_S$ , where  $x_i$  is the  $i$ th row of  $X$ .

## 2.2 *fastRG* samples from a generalization of the RDPG

Under the RDPG as described in Young and Scheinerman [2007], the expectation of the adjacency matrix is  $XX^T$  for some matrix  $X \in \mathbb{R}^{n \times K}$ . This implies that the expected adjacency matrix is always non-negative definite (i.e. its eigenvalues are non-negative). However, some parameterizations of the SBM (and other blockmodels) lead to an expected adjacency matrix with negative eigenvalues (i.e. it is not non-negative definite); for example, if the off-diagonal elements of  $S$  are larger than the diagonal elements, then  $XXS^T$  could have negative eigenvalues. Moreover, even if the elements of  $X$  and  $S$  are positive, as is the case for the low rank models in Table 1 and as is required for *fastRG*, it is still possible for  $XXS^T$  to have negative eigenvalues. By modifying the RDPG to incorporate a matrix  $S$ , the model class below incorporates all types of blockmodels.

**Definition 2.** [Generalized Random Product Graph (gRPG) model] For  $n$  nodes in  $K$  dimensions, the gRPG is parameterized by  $X \in \mathbb{R}^{n \times K}$  and  $S \in \mathbb{R}^{K \times K}$ , where each node  $i$  is assigned the  $i$ th row of  $X$ ,  $x_i = (X_{i1}, \dots, X_{iK})^T \in \mathbb{R}^K$ . For  $i, j \in V$ , define

$$\lambda_{ij} = \langle x_i, x_j \rangle_S = \sum_k \sum_l X_{ik} S_{kl} X_{jl}.$$

Under the gRPG, the adjacency matrix  $A \in \mathbb{R}^{n \times n}$  contains independent elements and the distribution of  $A_{ij}$  (i.e. the number of edge from  $i$  to  $j$ ) is fully parameterized by  $f(\lambda_{ij})$ , where  $f$  is some mean function.

Below, we will use the fact that the gRPG only requires that the  $\lambda_{ij}$ s specify the distribution of  $A_{ij}$ , allowing for  $A_{ij}$  to be non-binary (as in multi-graphs and weighted graphs) or to have edge probabilities which are a function of  $\lambda_{ij}$ .

**Theorem 2.** For  $X \in \mathbb{R}^{n \times K}$  and  $S \in \mathbb{R}^{K \times K}$ , each with non-negative elements, if  $\tilde{A}$  is the adjacency matrix of a graph sampled with  $\text{fastRG}(X, S)$ , then  $\tilde{A}$  is a Poisson gRPG with  $\tilde{A}_{ij} \sim \text{Poisson}(\langle x_i, x_j \rangle_S)$ .

The proof is contained in the appendix.

**Remark 1** (Simulating an undirected graph). As defined, both the gRPG model and  $\text{fastRG}$  generate directed graphs. An “undirected gRPG” should add a constraint to Definition 2 that  $A_{ij} = A_{ji}$  for all  $i, j$ . To sample such a graph with  $\text{fastRG}$ , input  $S/2$  instead of  $S$ , then after sampling a directed graph with  $\text{fastRG}$ , symmetrize each edge by removing its direction (this doubles the probability of an edge, hence the need to input  $S/2$ ). Theorem 2 can be easily extended to show this is an undirected gRPG.

**Remark 2** (Simulating a graph without self-loops). As defined, both the gRPG model and  $\text{fastRG}$  generate graphs with self-loops. A “gRPG without self-loops” should add a constraint to Definition 2 that  $A_{ii} = 0$  for all  $i$ . A graph from  $\text{fastRG}$  can be converted to a gRPG without self-loops by simply (1) sampling  $m \sim \text{Poisson}(\sum_{u,v} \tilde{S}_{uv} - \sum_i \langle x_i, x_i \rangle_S)$  and (2) resampling any edge that is a self-loop. The proof of Theorem 2 can be extended to show that this is equivalent.

### 2.3 Approximate Bernoulli-edges

To create a simple graph with  $\text{fastRG}$  (i.e. no multiple edges, no self-loops, and undirected), first sample a graph with  $\text{fastRG}$ . Then, perform the modifications described in Remarks 1 and 2. Then, keep an edge between  $i$  and  $j$  if there is at least one edge in the multiple edge graph; define the threshold function,  $t(A_{ij}) = \mathbb{1}(A_{ij} > 0)$ , where  $t(A)$  applies element-wise.

If  $\tilde{A}$  is a Poisson gRPG, then  $t(\tilde{A})$  is a Bernoulli gRPG with mean function  $f(\lambda_{ij}) = 1 - \exp(-\lambda_{ij})$ . Let  $B$  be distributed as Bernoulli gRPG( $X, S$ ) with identity mean function,  $B_{ij} \sim \text{Bernoulli}(\lambda_{ij})$ . Theorem 3 shows that in the sparse setting, there is a coupling between  $t(\tilde{A})$  and  $B$  such that  $t(\tilde{A})$  is approximately equal to  $B$ . The theorem is asymptotic in  $n$ ; a superscript of  $n$  is suppressed on  $\tilde{A}, B$  and  $\lambda$ .

**Theorem 3.** Let  $\tilde{A}$  be a Poisson gRPG and let  $B$  be a Bernoulli gRPG using the same set of  $\lambda_{ij}$ s, with  $\tilde{A}_{ij} \sim \text{Poisson}(\lambda_{ij})$  and  $B_{ij} \sim \text{Bernoulli}(\lambda_{ij})$ . Let  $t(\cdot)$  be the thresholding function for  $\tilde{A}$ .

Let  $\alpha_n$  be a sequence. If  $\lambda_{ij} = O(\alpha_n/n)$  for all  $i, j$  and there exists some constant  $c > 0$  and  $N > 0$  such that  $\sum_{ij} \lambda_{ij} > c\alpha_n n$  for all  $n > N$ , then there exists a coupling between  $t(\tilde{A})$  and  $B$  such that

$$\frac{E \|t(\tilde{A}) - B\|_F^2}{E \|B\|_F^2} = O(\alpha_n/n).$$

A proof is contained in the appendix.

For example, in the sparse graph setting where  $\lambda_{ij} = O(1/n)$  and  $\sum_{ij} \lambda_{ij} = O(n)$ ,  $\alpha_n = 1$ . Under this setting and the coupling defined in the proof, all of the  $O(n)$  edges in  $t(\tilde{A})$  are contained in  $B$  and  $B$  has an extra  $O(1)$  more edges than  $t(\tilde{A})$ .

## 2.4 Implementation of fastRG

Code at <https://github.com/karlrohe/fastRG> gives an implementation of fastRG in R. It also provides wrappers that simulate the SBM, Degree Corrected SBM, Overlapping SBM, and Mixed Membership SBM. The code for these models first generates the appropriate  $X$  and then calls fastRG. In order to help control the edge density of the graph, fastRG and its wrappers can be given an additional argument avgDeg. If avgDeg is given, then the matrix  $S$  is scaled so that fastRG simulates a graph with expected average degree equal to avgDeg. Without this, parameterizations can easily produce very dense graphs.

To accelerate the running time of fastRG, the implementation is slightly different than the statement of the algorithm above. The difference can be thought of as sampling all of the  $(U, V)$  pairs before sampling any of the  $I$ s or  $J$ s. In particular, the implementation samples  $\bar{\omega} \in R^{K \times K}$  as  $\text{multinomial}(m, \tilde{S} / \sum_{u,v} \tilde{S}_{uv})$ . Then, for each  $u \in \{1, \dots, K\}$ , it samples  $\sum_v \bar{\omega}_{u,v}$ -many  $I$ s from the distribution  $\tilde{X}_u$ . Similarly, for each  $v \in \{1, \dots, K\}$ , it samples  $\sum_u \bar{\omega}_{u,v}$ -many  $J$ s from the distribution  $\tilde{X}_v$ . Finally, the indexes are appropriately arranged so that there are  $\bar{\omega}_{u,v}$ -many edges  $(I, J)$  where  $I \sim X_u$  and  $J \sim X_v$ . Recall that the statement of fastRG above allows for  $X$  and  $Y$ , where those matrices can have different numbers of rows and/or columns; the implementation also allows for this.

Under the SBM, it is possible to use fastRG to sample from the Bernoulli gRPG with the *identity* mean function instead of the mean function  $1 - \exp(-\langle x_i, x_j \rangle_S)$  that is created by the thresholding function  $t$  from Section 2.3. The wrapper for the SBM does this by first transforming each element of  $S$  as  $-\ln(1 - S_{ij})$  and then calling fastRG. The others models are not amenable to this trick; by default, they sample from the Poisson gRPG with identity mean function.

## 2.5 Experiments

To examine the running time of fastRG, we simulated a range of different values of  $n$  and  $E(m)$ , where  $E(m)$  is the expected number of edges. In all simulations  $X = Y$  and  $K = 5$ . The elements of  $X$  are independent  $\text{Poisson}(1)$  random variables and the elements of  $S$  are independent  $\text{Uniform}[0, 1]$  random variables. To specify  $E(m)$ , the parameter avgDeg is set to  $E(m)/n$ . The values of  $n$  range from 10,000 to 10,000,000 and the values of  $E(m)$  range from 100,000 to 100,000,000. The graph was taken to be directed, with self-loops and multiple edges. Moreover, the reported times are only to generate the edge list of the random graph; the edge list is not converted into a sparse adjacency matrix, which in some experiments would have more than doubled the running time. Each pair of  $n$  and  $E(m)$  is simulated one time; deviations around the trend lines indicate the variability in run time.

In Figure 1, the vertical axes present the running time in R on a Retina 5K iMac, 27-inch, Late 2014 with 3.5 GHz Intel i5 and 8GB of 1600 MHz DDR3 memory. In the left panel of Figure 1, each line corresponds to a single value of  $n$  and  $E(m)$  increases along the horizontal axis. In the right panel of Figure 1, each line corresponds to a single value of  $E(m)$  and  $n$  increases along the horizontal axis. All axes are on the  $\log_{10}$  scale. The solid black line has a slope of 1. Because the data aligns with this black line, this suggests that fastRG runs in linear time.

The computational bottleneck is sampling the  $I$ s and  $J$ s. The implementation uses Walker’s Alias Method [Walker, 1977] (via `sample` in R). To take  $m$  samples from a distribution over  $n$  elements, Walker’s Alias Method requires  $O(m + \ln(n)n)$  operations [Vose, 1991]. However, the log dependence is not clearly evident in the right plot of Figure 1; perhaps it would be visible for larger values of  $n$ .

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## A Proofs

For an integer  $d$ , define  $1_d \in \mathbb{R}^d$  as a vector of ones. The proof of Theorem 2 requires the following lemma, which says that a vector (or matrix) of independent Poisson entries becomes multinomial when you condition on the sum of the vector (or matrix).

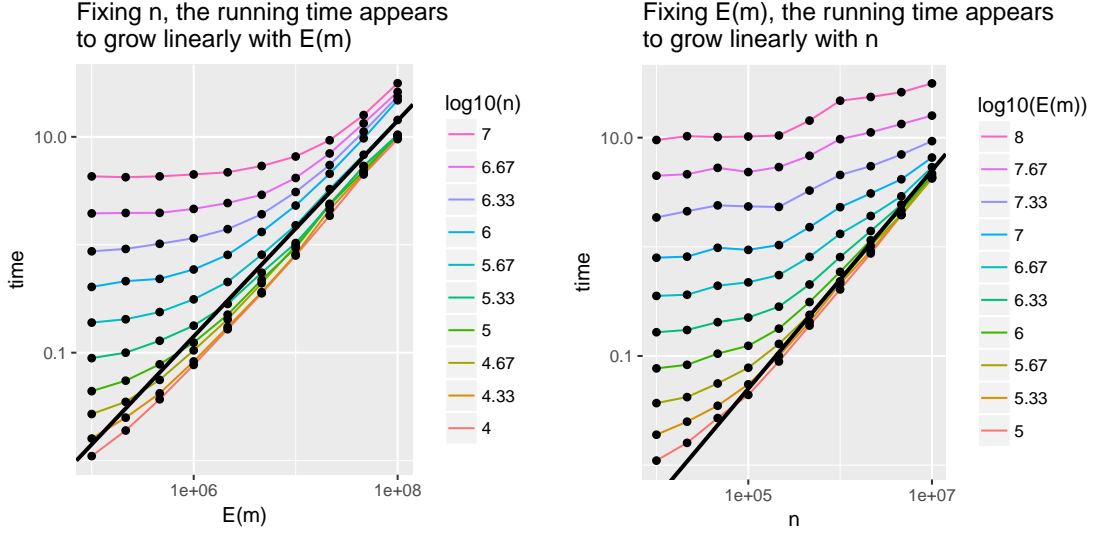


Figure 1: Both plots present the same experimental data. In the left plot, each line corresponds to a different value of  $n$  and they are presented as a function of  $E(m)$ . In the right plot, each line corresponds to a different value of  $E(m)$  and they are presented as a function of  $n$ . On the right side of both plots, the lines start to align with the solid black line, suggesting a linear dependence on  $E(m)$  and  $n$ .

**Lemma 1.** Let  $A \in \mathbb{R}^{n \times n}$  be the random matrix whose  $i, j$ th element  $A_{ij} \stackrel{i.d.}{\sim} \text{Pois}(\lambda_{ij})$   $i, j = 1, \dots, n$ . Then conditioned on  $\sum_{i,j} A_{ij} = 1_n^T A 1_n = m$ ,

$$(A_{11}, A_{12}, \dots, A_{nn}) \sim \text{Multinomial}(m, \lambda / \sum_{ij} \lambda_{ij})$$

where  $\lambda = (\lambda_{11}, \lambda_{12}, \dots, \lambda_{nn})$ . That is, let  $a \in \mathbb{R}^{n \times n}$  be a fixed matrix of integers with  $1_n^T a 1_n = m$ , then

$$\begin{aligned} \mathbb{P}(A = a | 1_n^T A 1_n = m) &= \mathbb{P}(A_{11} = a_{11}, A_{12} = a_{12}, \dots, A_{nn} = a_{nn} | 1_n^T A 1_n = m) \\ &= \frac{m!}{\prod_{i,j} a_{ij}!} \prod_{i,j} \left( \frac{\lambda_{ij}}{\lambda_{11} + \lambda_{12} + \dots + \lambda_{nn}} \right)^{a_{ij}}. \end{aligned}$$

For completeness, a proof of this classical result is given at the end of the paper. The next proof is a proof of Theorem 2.

*Proof.* Let  $A$  come from the Poisson gRPG with  $X$  and  $S$  and identity mean function. Let  $\tilde{A}$  be a sample from fastRG. For any fixed adjacency matrix  $a$ , we will show that  $\mathbb{P}(A = a) = \mathbb{P}(\tilde{A} = a)$ .

Define  $m = 1_n^T a 1_n$  and decompose the probabilities,

$$\mathbb{P}(A = a) = \mathbb{P}(1_n^T A 1_n = m) \mathbb{P}(A = a | 1_n^T A 1_n = m) \quad (1)$$

$$\mathbb{P}(\tilde{A} = a) = \mathbb{P}(1_n^T \tilde{A} 1_n = m) \mathbb{P}(\tilde{A} = a | 1_n^T \tilde{A} 1_n = m). \quad (2)$$

The proof will be divided into two parts. The first part shows that  $\mathbb{P}(1_n^T A 1_n = m) = \mathbb{P}(1_n^T \tilde{A} 1_n = m)$  and the second part will show that  $\mathbb{P}(A = a | 1_n^T A 1_n = m) = \mathbb{P}(\tilde{A} = a | 1_n^T \tilde{A} 1_n = m)$ .

**Part 1:** Because the sum of independent Poisson variables is still Poisson,  $\sum_{ij} A_{ij} \sim \text{Poisson}(\sum_{ij} \lambda_{ij})$ . So, we must only show that  $1_n^T A 1_n$  and  $1_n^T \tilde{A} 1_n$  have the same Poisson parameter:

$$\sum_{ij} \lambda_{ij} = 1_n^T X S X 1_n = 1_n^T X C C^{-1} S C^{-1} C X 1_n = 1_n^T \tilde{X} \tilde{S} \tilde{X} 1_n = 1_n^T \tilde{X} \tilde{S} \tilde{X} 1_n = 1_K^T \tilde{S} 1_K = \sum_{u,v} \tilde{S}_{uv}.$$

**Part 2:** After conditioning on  $1_n^T A 1_n = m$ , Lemma 1 shows that  $A$  has the multinomial distribution. In fastRG, we first sample  $1_n^T \tilde{A} 1_n$  and then add edges with the multinomial distribution. So, we must only

show that the multinomial edge probabilities are equal for  $A$  and  $\tilde{A}$ . From Lemma 1, the multinomial edge probabilities for  $A$  are  $\lambda_{ij}/\sum_{a,b}\lambda_{ab}$ . To compute the multinomial edge probabilities for  $\tilde{A}$ , recall that  $(I, J)$  is a single edge added to the graph in fastRG. By Theorem 1,

$$\mathbb{P}(\tilde{A}_{ij} = 1 | 1_n^T \tilde{A} 1_n = 1) = \mathbb{P}((I, J) = (i, j)) = \frac{\langle x_i, x_j \rangle_S}{\sum_{a,b} \langle x_a, x_b \rangle_S} = \frac{\lambda_{ij}}{\sum_{a,b} \lambda_{ab}}$$

This concludes the proof.  $\square$

*Proof of Theorem 3.* Let  $U_{ij} \stackrel{i.i.d}{\sim} \text{Uniform}(0, 1)$ . Define  $\mathcal{A}$  and  $\mathcal{B}$ :

$$\begin{aligned}\mathcal{A}_{ij} &= \mathbf{1}(U_{ij} > 1 - e^{-\lambda_{ij}}), \\ \mathcal{B}_{ij} &= \mathbf{1}(U_{ij} > \lambda_{ij}).\end{aligned}$$

Note that  $\mathcal{A}$  and  $\mathcal{B}$  are equal in distribution to  $t(\tilde{A})$  and  $B$  respectively. By Taylor expansion,

$$E\|\mathcal{A} - \mathcal{B}\|_F^2 = \sum_{i,j} (\lambda_{ij} - (1 - e^{-\lambda_{ij}})) = \sum_{i,j} \sum_{k=2}^{\infty} (-\lambda_{ij})^k / k! = \sum_{i,j} O(\lambda_{ij}^2) = \sum_{i,j} O((\alpha_n/n)^2) = O(\alpha_n^2).$$

Then,  $E\|B\|_F^2 = \sum_{i,j} \lambda_{ij} > c\alpha_n n$ . So, defining  $t(\tilde{A})$  and  $B$  with the above coupling yields the result.  $\square$

*proof of Lemma 1.*

$$\begin{aligned}\mathbb{P}(A = a | 1_n^T A 1_n = m) &= \frac{\mathbb{P}(A = a)}{\mathbb{P}(1_n^T A 1_n = m)} = \frac{\prod_{i,j} \frac{\lambda_{ij}^{a_{ij}}}{a_{ij}!} e^{-\lambda_{ij}}}{\frac{(\lambda_{11} + \lambda_{12} + \dots + \lambda_{nn})^m}{m!} e^{-(\lambda_{11} + \lambda_{12} + \dots + \lambda_{nn})}} \\ &= \frac{m!}{\prod_{i,j} a_{ij}!} \prod_{i,j} \left( \frac{\lambda_{ij}}{\lambda_{11} + \lambda_{12} + \dots + \lambda_{nn}} \right)^{a_{ij}}\end{aligned}$$

$\square$

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